Higher approximations in boundary-layer theory

Part 2. Application to leading edges

By MILTON VAN DYKE

Department of Aeronautics and Astronautics, Stanford University

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The general analysis of Part 1 is applied to the calculation of the second-order viscous and thermal boundary layers for the axisymmetric stagnation point, unsymmetric plane stagnation point, and cusped leading edge at ideal incidence. The second-order effect upon heat transfer is found usually to be of the order of one-third of that for skin friction.

1. Introduction

A systematic procedure was developed in part 1 (Van Dyke 1962) for improving upon Prandtl's boundary-layer theory. The second approximation was studied in detail for steady laminar flow of a constant-property fluid past an analytic semi-infinite plane or axisymmetric body free of separation. Five additive second-order effects were identified.

Here that analysis is illustrated by application to specific problems. It is natural to consider first the Falkner–Skan family of self-similar flows, which require numerical integration of only ordinary differential equations. We choose the three most useful cases, corresponding to the plane and axisymmetric stagnation point and the flat plate (the first-order solutions being associated with the names of Hiemenz, Homann, and Blasius). Each of these can be taken as the basis of a Blasius series to extend the solution downstream over a body of general shape, and we adopt that point of view, which is in fact essential in the second approximation for unsymmetric plane flow. The body is taken to be either insulated or at a prescribed temperature (which may vary with distance).

Reference to an equation in Part 1 is made by giving its number preceded by the Roman numeral I.

2. Axisymmetric stagnation point

Consider axisymmetric flow past an analytic body whose nose radius L is taken as the unit of length. Then the (convex) curvature, radius, and angle of its surface are given by

$$\kappa(s) = 1+O(s^2), \tag{2.1a}$$

$$r_0(s) = s-\tfrac{1}{6}s^3+O(s^5), \tag{2.1b}$$

$$\cos\theta(s) = s+O(s^3). \tag{2.1c}$$

Suppose that the problem (I, 3.11) for the basic inviscid flow has been solved subject to appropriate upstream conditions. From that solution the surface speed (referred to some reference speed U_r) can be found as

$$U_1(s,0) = U_{11}s+O(s^3). \tag{2.2}$$

2.1. *First-order velocity and temperature*

We summarize the first-order theory, which is given in detail by Frössling (1940) and Schlichting (1960). The first-order stream function of (I, 4.8a) is expanded in the Blasius series

$$\psi_1(s, N) = (\tfrac{1}{2}U_{11})^{\frac{1}{2}}\, s^2[f_1(\eta) + O(s^2)], \tag{2.3a}$$

where

$$\eta = (2U_{11})^{\frac{1}{2}}\, N. \tag{2.3b}$$

Here and in the subsequent examples we normalize the independent and dependent variables according to the usual Falkner–Skan notation, because it leads to the simplest forms of the equations, particularly in the second approximation.

Substituting into the first-order problem (I, 4.9) for the velocity field gives

$$2f_1''' + 2f_1 f_1'' - f_1'^2 = -1, \tag{2.4a}$$

$$f_1(0) = f_1'(0) = 0, \quad f_1'(\infty) = 1. \tag{2.4b}$$

Numerical integration yields

$$f_1''(0) = 0{\cdot}927680 \tag{2.5a}$$

(of which Frössling 1940 gives the first four figures), and for large η

$$f_1(\eta) \sim \eta - \beta_1, \quad \beta_1 = 0{\cdot}80455. \tag{2.5b}$$

In the energy equation (I, 4.10a) the dissipative terms are $O(s^2)$. Hence an insulated body has no thermal boundary layer near the stagnation point, the surface temperature being the free-stream stagnation temperature on the axis, $H_1(0)$. On the other hand, suppose that the surface temperature (referred to some reference temperature T_r) is prescribed as

$$T_w(s) = T_{w0} + O(s^2). \tag{2.6}$$

Then expanding the temperature in a Blasius series as

$$t_1(s, N) = H_1(0) + [T_{w0} - H_1(0)]\, g_1(\eta) + O(s^2) \tag{2.7}$$

and substituting into (I, 4.10) gives

$$\sigma^{-1} g_1'' + f_1 g_1' = 0, \quad g_1(0) = 1, \quad g_1(\infty) = 0. \tag{2.8}$$

We consider values of the Prandtl number σ of 1·0 (in which case g_1 is known to be given in terms of f_1 by quadratures) and 0·7. Numerical integration (confirmed to four figures by Frössling 1940) yields

$$-g_1'(0) \begin{cases} = 0{\cdot}53898, & \sigma = 1{\cdot}0, \\ = 0{\cdot}47050, & \sigma = 0{\cdot}7. \end{cases} \tag{2.9}$$

2.2. *Second-order velocity*

Four of the five second-order effects are local, and can be calculated for the stagnation point independent of the subsequent flow. However, the displacement speed can be found only by calculating the first-order boundary layer over the entire body, and then solving the second-order outer problem (I, 3.12), (I, 3.27 *b*) for the flow due to displacement thickness. Suppose that this has been done, giving at the surface the second-order increment

$$U_2(s,0) = U_{21}s + O(s^3). \tag{2.10}$$

We expand the second-order stream function in a Blasius series like that (2.3) for the first approximation, which reduces to universal functions if taken in the form

$$\psi_2(s,N) = s^2[F_{1c}(\eta) + (2U_{11})^{-1}B_1'(0)F_{1v}(\eta) + (2U_{11})^{-\frac{1}{2}}U_{21}F_{1d}(\eta)] + O(s^4). \tag{2.11}$$

Here F_{1c}, F_{1v}, and F_{1d} represent the effects of curvature (longitudinal and transverse combined), external vorticity, and displacement speed, where $B_1'(0)$ is the dimensionless value of ω/r on the axis, ω being the vorticity in the oncoming stream.

The integral appearing in the momentum equation (I, 4.11 *a*) can be evaluated using the fact that the general Falkner–Skan equation

$$f''' + ff'' + \beta(1-f'^2) = 0, \tag{2.12a}$$

of which (2.4*a*) is the special case $\beta = \frac{1}{2}$, can be written

$$(1+\beta)f'^2 = (f'' + ff' + \beta\eta)'. \tag{2.12b}$$

Thus the second-order problem (I, 4.11) becomes, using (2.4)

$$F_{1c}''' + f_1F_{1c}'' - f_1'F_{1c}' + f_1''F_{1c} = \eta(f_1f_1'' - \tfrac{1}{2}f_1'^2 + \tfrac{1}{6}) + \tfrac{1}{3}(f_1'' + f_1f_1' + \beta_1); \tag{2.13a}$$

$$F_{1c}(0) = F_{1c}'(0) = 0, \quad F_{1c}'(\infty) = 0; \tag{2.13b}$$

$$F_{1v}''' + f_1F_{1v}'' - f_1'F_{1v}' + f_1''F_{1v} = -\beta_1, \tag{2.14a}$$

$$F_{1v}(0) = F_{1v}'(0) = 0, \quad F_{1v}''(\infty) = 1; \tag{2.14b}$$

$$F_{1d}''' + f_1F_{1d}'' - f_1'F_{1d}' + f_1''F_{1d} = -1, \tag{2.15a}$$

$$F_{1d}(0) = F_{1d}'(0) = 0, \quad F_{1d}'(\infty) = 1. \tag{2.15b}$$

The solution of the displacement-speed problem (2.15) is given in terms of the first-order function f_1 by

$$F_{1d}(\eta) = \tfrac{1}{2}(f_1 + \eta f_1'). \tag{2.16}$$

N. Rott has pointed out to the author that this relationship, like others to be encountered later, is a straightforward consequence of the fact that the displacement speed is locally similar to the basic inviscid surface speed. The problems for curvature (2.13) and vorticity (2.14) have been integrated numerically, with the result that

$$\left.\begin{aligned} F_{1c}''(0) &= -0{\cdot}62260,\\ F_{1v}''(0) &= 1{\cdot}76861,\\ F_{1d}''(0) &= 1{\cdot}39152. \end{aligned}\right\} \tag{2.17}$$

The first and third of these numbers agree (aside from a different normalization) with those given by Lenard (1961). The second number disagrees with that found by Rott & Lenard (1959), Lenard (1961), and Maslen (1962), because they do not include the second-order change in pressure induced upon the boundary layer by interaction of the displacement thickness with the external vorticity (cf. (I, 3.14*a*)). Thus they solve (2.14*a*) with the non-homogeneous term $-\beta_1$

omitted. Because a particular integral of (2.14*a*) is given by $\frac{1}{2}\beta_1(f_1+\eta f_1')$, one readily confirms that their value is $F''_{1v}(0) - \frac{3}{2}\beta_1 f''_1(0)$ = O-64906.

Kemp (1959) has found an exact solution of the Navier-Stokes equations that generalizes **Homann's** solution to include external vorticity. For weak vorticity it reduces to the perturbation solutions just mentioned, because Kemp also disregards the pressure change. It can be shown that his solution is one member of a family of exact solutions, another of which reduces to the present solution that includes the pressure change.

Thus previous investigators have calculated only the kinematic effects of external vorticity. As discussed in Part 1, this is acceptable provided the remaining effect is included by considering displacement pressure rather than displacement speed. Lenard (1961) and Maslen (1962) have treated the effect of displacement as well as vorticity, but they unfortunately consider displacement speed, so that their net result is incorrect. (Lenard has corrected this matter, taking the point of view of displacement pressure, in an appendix to the revised version of his thesis.)

2.3. Second-order temperature

As in the first approximation, all compressibility terms (proportional to m^2) in the energy equation are $O(s^2)$. Hence an insulated stagnation point has no thermal boundary layer to second order. If instead the surface temperature is prescribed by (2.6), universal functions are obtained by writing the Blasius series for t_2 as

$$t_2(s,N) = (2/U_{11})^{\frac{1}{2}}[T_{w0} - H_1(0)] \times [G_{1c}(\eta) + (2U_{11})^{-1}B_1'(0)\,G_{1v}(\eta) + (2U_{11})^{-\frac{1}{2}}U_{21}\,G_{1d}(\eta)] + O(s^2). \quad \textbf{(2.18)}$$

Substituting into (**I, 4.12**) and using (2.8) and (2.16) gives

$$\sigma^{-1}G''_{1c} + f_1\,G'_{1c} = (\eta f_1 - \sigma^{-1} - F_{1c})\,g;, \quad G_{1c}(0) = G_{1c}(\infty) = 0; \qquad (2.19)$$

$$\sigma^{-1}G''_{1v} + f_1\,G'_{1v} = -\,g_1'F_{1v}, \quad G_{1v}(0) = G_{1v}(\infty) = 0; \qquad (2.20)$$

$$\sigma^{-1}G''_{1d} + f_1\,G'_{1d} = -\,\tfrac{1}{2}(f_1 + \eta f_1')\,g_1', \quad G_{1d}(0) = G_{1d}(\infty) = 0. \qquad (2.21)$$

The solution of the displacement-speed problem (2.2 **l**) is given by

$$G_{1d}(\eta) = \tfrac{1}{2}\eta g_1'(\eta). \qquad (2.22)$$

The other two problems have been integrated numerically. Thus

$$\left.\begin{aligned} \text{-G;,(O)} &= \text{O-48296}, \quad \sigma = 1.0,\\ &= \text{o-50194}, \quad \sigma = 0.7;\\ \text{-G;,(O)} &= \text{O-42073}, \quad \sigma = 1.0,\\ &= 0.38270, \quad \sigma = 0.7;\\ -G'_{1d}(0) &= 0.26949, \quad \sigma = 1.0,\\ &= 0.23525, \quad \sigma = 0.7. \end{aligned}\right| \qquad (2.23)$$

Because he neglects the induced pressure gradient, Kemp (1959) finds for G_{1v} the above values plus the solution of

$$\sigma^{-1}\Delta'' + f_1\Delta' = \tfrac{1}{2}\beta_1(f_1 + \eta f_1')\,g_1', \quad \text{A}(0) = \Delta(\infty) = 0. \qquad \textbf{(2.24a)}$$

The solution is given by $\Delta = -\frac{1}{2}\beta_1 \eta g_1'$, so that his result corresponds to values of $-G_{1v}'(0)$ of

$$\left.\begin{matrix}0\cdot42073\\0\cdot38270\end{matrix}\right\} + \tfrac{1}{2}\beta_1 g_1'(0) = \left\{\begin{matrix}0\cdot20391, & \sigma = 1\cdot0,\\ 0\cdot19343, & \sigma = 0\cdot7.\end{matrix}\right\} \tag{2.24b}$$

2.4. *Skin friction and heat transfer*

For convenience we present results in physical terms; that is, in this section only all symbols denote actual dimensional quantities. Substituting the preceding results into (I, 4.13) gives for the skin friction

$$\tau = \rho\nu^{\frac{1}{2}}U_{11}^{\frac{3}{2}}s[1\cdot311938 - 1\cdot24520(\nu/U_{11}L^2)^{\frac{1}{2}} - 1\cdot76861(\nu/U_{11}^3)^{\frac{1}{2}}(\omega/r)_0 + 1\cdot96791\nu^{\frac{1}{2}}U_{21}/U_{11} + O(\nu)] + O(s^3). \tag{2.25}$$

Here L is the nose radius, s the actual distance from the stagnation point, and $(\omega/r)_0$ the ratio of vorticity in the oncoming stream to the radius, evaluated on the axis. The surface speed in the outer flow is $(U_{11} + \nu^{\frac{1}{2}}U_{21} + \ldots)s + O(s^3)$, U_{21} representing the flow due to displacement thickness. The first term in (2.25) is Homann's result of classical boundary-layer theory, followed by the second-order corrections for curvature, external vorticity, and displacement speed.

Convex curvature reduces the skin friction. External vorticity contributes

$$\tau_{2v} = -1\cdot76861\mu\omega_0. \tag{2.26}$$

Here the numerical factor shows that through interaction with the boundary layer the shear associated with external vorticity is increased 77 % above its value just outside the boundary layer. Neglecting the pressure gradient associated with external vorticity would reduce this factor to $-0\cdot64906$, which is the result of Rott & Lenard (1959) and Kemp (1959). Kemp uses a vorticity parameter

$$\Omega = -(\nu/2U_{11}^3)^{\frac{1}{2}}(\omega/r)_0 \tag{2.27}$$

and finds as the relative correction to skin friction an average value of $1 + 0\cdot78\Omega$ for $0 \leqslant \Omega \leqslant 0\cdot6$; from (2.25) the present result for small Ω is $1 + 1\cdot90648\Omega$.

Substituting into (I, 4.14) gives for the heat transfer to the body, in terms of dimensional quantities,

$$\frac{q}{k(U_{11}/\nu)^{\frac{1}{2}}(T_{w0}-T_0)} = \begin{Bmatrix}0\cdot76223\\0\cdot66538\end{Bmatrix} + \begin{Bmatrix}0\cdot96592\\1\cdot00388\end{Bmatrix}(\nu/U_{11}L^2)^{\frac{1}{2}} - \begin{Bmatrix}0\cdot42073\\0\cdot38270\end{Bmatrix}(\nu/U_{11}^3)^{\frac{1}{2}}(\omega/r)_0 + \begin{Bmatrix}0\cdot38112\\0\cdot33269\end{Bmatrix}\nu^{\frac{1}{2}}U_{21}/U_{11} + O(\nu, s^2), \quad \sigma = \begin{Bmatrix}1\cdot0,\\0\cdot7.\end{Bmatrix} \tag{2.28}$$

Here T_0 is the stagnation temperature on the axis, and T_{w0} the temperature of the body at its nose. Each second-order correction has the same sign for the heat transfer as for the skin friction except for curvature, where the decrease due to longitudinal curvature is overbalanced by an increase due to transverse curvature. Except for that term, the relative effect is about one-third as great for heat transfer as for skin friction. The relative correction factor for the effect of external vorticity is

$$1+\begin{Bmatrix}0{\cdot}7806\\0{\cdot}8134\end{Bmatrix}\Omega,\quad \sigma=\begin{Bmatrix}1{\cdot}0,\\0{\cdot}7,\end{Bmatrix} \tag{2.29}$$

rather than the $1+(0{\cdot}3783$ or $0{\cdot}4111)\,\Omega$ that results from neglecting the induced pressure.

3. Plane stagnation point

For symmetric flow the plane stagnation point is treated like the axisymmetric one. Asymmetry introduces novel features, however. The (dimensionless) inviscid surface speed has then the form

$$U_1(s,0) = U_{11}s+U_{12}s^2+O(s^3); \tag{3.1}$$

and if the radius of the body at the inviscid stagnation point is taken as the reference length L, the curvature is

$$\kappa(s) = 1+O(s). \tag{3.2}$$

3.1. *First-order velocity and temperature*

The Blasius–Howarth series for the first-order stream function is (Schlichting 1960)

$$\psi_1(s,N) = U_{11}^{\frac{1}{2}}s[f_1(\eta)+3(U_{12}/U_{11})\,sf_2(\eta)+O(s^2)], \tag{3.3a}$$

where

$$\eta = U_{11}^{\frac{1}{2}}N. \tag{3.3b}$$

Two terms must be kept here to find one in the second approximation. Substituting into (I, 4.9) gives

$$f_1'''+f_1f_1''-f_1'^2 = -1,\quad f_1(0)=f_1'(0)=0,\quad f_1'(\infty)=1; \tag{3.4a}$$

$$f_2'''+f_1f_2''-3f_1'f_2'+2f_2''f_2 = -1,\quad f_2(0)=f_2'(0)=0,\quad f_2'(\infty)=\tfrac{1}{3}. \tag{3.4b}$$

Numerical integration yields

$$\left.\begin{aligned}f_1''(0) &= 1{\cdot}232588;\quad f_1(\eta)\sim\eta-\beta_1,\quad \beta_1 = 0{\cdot}647900;\\ f_2''(0) &= 0{\cdot}798744;\quad f_2(\eta)\sim\tfrac{1}{3}\eta-\beta_2,\quad \beta_2 = 0{\cdot}0270.\end{aligned}\right\} \tag{3.5}$$

Again there is no thermal boundary layer near the stagnation point if the body is insulated. If its temperature is prescribed according to

$$T_w(s) = T_{w0}+T_{w1}s+O(s^2) \tag{3.6}$$

the Blasius series for temperature is (Frössling 1940)

$$t_1(s,N) = H_1(0)+[T_{w0}-H_1(0)]\,[g_1(\eta)+(U_{12}/U_{11})\,s\,g_2(\eta)]+T_{w1}s\,k_2(\eta)+O(s^2), \tag{3.7}$$

where substituting into (I, 4.10) gives

$$\sigma^{-1}g_1''+f_1g_1' = 0,\qquad g_1(0)=1,\quad g_1(\infty)=0; \tag{3.8a}$$

$$\sigma^{-1}g_2''+f_1g_2'-f_1'g_2 = -6f_2g_1',\quad g_2(0)=g_2(\infty)=0; \tag{3.8b}$$

$$\sigma^{-1}k_2''+f_1k_2'-f_1'k_2 = 0,\qquad k_2(0)=1,\quad k_2(\infty)=0. \tag{3.8c}$$

For $\sigma = 0{\cdot}7$, numerical integration yields

$$\left.\begin{aligned} -g_1'(0) &= 0{\cdot}495867, \\ -g_2'(0) &= 0{\cdot}362253, \\ -k_2'(0) &= 0{\cdot}708981. \end{aligned}\right\} \tag{3.9}$$

3.2. *Second-order velocity*

Suppose that the flow due to displacement thickness has been calculated to find

$$U_2(s,0) = U_{20} + U_{21}s + O(s^2). \tag{3.10}$$

The second-order stream function is then reduced to universal functions by setting

$$\begin{aligned} \psi_2(s,N) = sF_{1c}(\eta) + U_{11}^{-1}B_1'(0)\,[F_{0v}(\eta) + 6(U_{12}/U_{11})\,sF_{1v}(\eta)] + U_{11}^{-\frac{1}{2}}U_{20}F_{0d}(\eta) \\ + U_{11}^{-\frac{1}{2}}s[U_{21}F_{1d}(\eta) + 2(U_{12}U_{20}/U_{11})\,E_{1d}(\eta)] + O(s^2). \end{aligned} \tag{3.11}$$

If the flow is symmetric, only odd powers of s appear (and moreover the external-vorticity parameter $B_1'(0)$ vanishes), but asymmetry introduces terms of order unity associated with external vorticity and displacement speed. Substituting into (I, 4.11) gives for these leading terms

$$F_{0v}''' + f_1F_{0v}'' - f_1'F_{0v}' = -\beta_1, \quad F_{0v}(0) = F_{0v}'(0) = 0, \quad F_{0v}''(\infty) = 1; \tag{3.12}$$

$$F_{0d}''' + f_1F_{0d}'' - f_1'F_{0d}' = -1, \quad F_{0d}(0) = F_{0d}'(0) = 0, \quad F_{0d}'(\infty) = 1. \tag{3.13}$$

The solution of the displacement-speed problem (3.13) is given by

$$F_{0d}(\eta) = f_1(\eta) \tag{3.14}$$

and this has a simple interpretation. The inviscid surface speed is, from (I, 3.2), (3.1), and (3.10),

$$U_1(s,0) + R^{-\frac{1}{2}}U_2(s,0) + \ldots = R^{-\frac{1}{2}}U_{20} + s(U_{11} + R^{-\frac{1}{2}}U_{21} + \ldots) + O(s^2) \tag{3.15}$$

so that the displacement effect of the first-order boundary layer shifts the stagnation point from $s = 0$ to $s = -R^{-\frac{1}{2}}U_{20}/U_{11} + \ldots$. The second-order term F_{0d} simply serves to shift the origin of the Hiemenz solution to that point. Numerical integration of the equation (3.12) for F_{0v} is simplified by the fact that a particular integral is given by $\beta_1 f_1(\eta)$. Thus

$$\left.\begin{aligned} F_{0v}''(0) &= 1{\cdot}40652, \\ F_{0d}''(0) &= 1{\cdot}232588. \end{aligned}\right\} \tag{3.16}$$

The vorticity problem (3.12) was first solved by Stuart (1959), who showed that (like Kemp's solution for the axisymmetric problem) it in fact represents an exact solution of the Navier–Stokes equations. Stuart actually solves the homogeneous equation, which from our point of view gives only the kinematic effect of external vorticity. However, he noted that this solution can be generalized by shifting the stagnation point; this re-introduces β_1 into the equation, and because $\beta_1 f_1$ is then a particular integral, one reproduces his value of $g'(0)$ as $F_{0v}''(0) - \beta_1 f_1''(0) = 0{\cdot}60793$.

For the terms proportional to s the problems are

$$F'''_{1c}+f_1F''_{1c}-2f'_1F'_{1c}+f''_1F_{1c} = \eta(f_1f''_1-f'^2_1+2)+\beta_1,$$

$$F,,(O) = F;;,(O) = 0,\ F''_{1c}(\infty) = -1; \quad (3.17)$$

$$F'''_{1v}+f_1F''_{1v}-2f'_1F'_{1v}+f''_1F_{1v} = f'_2F'_{0v}-f_2F''_{0v}-\beta_2,$$

$$F_{1v}(0) = F'_{1v}(0) = F'_{1v}(\infty) = 0; \quad (3.18)$$

$$F'''_{1d}+f_1F''_{1d}-2f'_1F'_{1d}+f''_1F_{1d} = -2,\ F_{1d}(0) = F'_{1d}(0) = 0,\ F'_{1d}(\infty) = 1; \quad (3.19)$$

$$E'''_{1d}+f_1E''_{1d}-2f'_1E'_{1d}+f''_1E_{1d} = 3(f'_1f'_2-f''_1f_2)-1,$$

$$E_{1d}(0) = E'_{1d}(0) = E'_{1d}(\infty) = 0. \quad (3.20)$$

The solutions of (3.19) and (3.20) are given by

$$F_{1d}(\eta) = \tfrac{1}{2}(f_1+\eta f'_1), \quad (3.21a)$$

$$E_{1d}(\eta) = 3f_2-\tfrac{1}{2}(f_1+\eta f'_1). \quad (3.21b)$$

The other two problems have been solved numerically, with the result that

$$\left.\begin{aligned} F''_{1c}(0) &= -1.913255,\\ F''_{1v}(0) &= -0.06299,\\ F''_{1d}(0) &= 1.848882,\\ E''_{1d}(0) &= 0.547350. \end{aligned}\right\} \quad (3.22)$$

The first and third of these agree to five figures with the values found by Lenard (1961).

3.3. *Second-order temperature*

Compressibility effects in the energy equation are $O(s^2)$ for symmetric flow and O(s) for unsymmetric flow, but in any case negligible near the stagnation point. Hence an insulated body has no thermal boundary layer there to second order. For a body with temperature prescribed by (3.6), set

$$\begin{aligned} t_2(s,N) = {} & U_{11}^{-\frac{1}{2}}[T_{w0}-H_1(0)]\,[G_{1c}(\eta)+(U_{12}/U_{11}^2)\,B'_1(0)\,G_{1v}(\eta) \\ & +U_{11}^{-\frac{1}{2}}U_{21}G_{1d}(\eta)+U_{12}U_{20}U_{11}^{-\frac{3}{2}}J_{1d}(\eta)] \\ & +T_{w1}U_{11}^{-1}[U_{11}^{-\frac{1}{2}}B'_1(0)\,K_{1v}(\eta)+U_{20}K_{1d}(\eta)]+O(s). \end{aligned} \quad (3.23)$$

Substituting into (I, 4.12) gives the problems

$$\sigma^{-1}G''_{1c}+f_1G'_{1c} = (\eta f_1-\sigma^{-1}-F_{1c})\,g'_1, \qquad G_{1c}(0) = G_{1c}(\infty) = 0; \quad (3.24)$$

$$\sigma^{-1}G''_{1v}\text{-}t f_1G'_{1v} = g_2F'_{0v}-6g'_1F_{1v}, \qquad G_{1v}(0) = G_{1v}(\infty) = 0; \quad (3.25)$$

$$\sigma^{-1}G''_{1d}+f_1G'_{1d} = -\tfrac{1}{2}(f_1+\eta f'_1)\,g'_1, \qquad G_{1d}(0) = G_{1d}(\infty) = 0; \quad (3.26)$$

$$\sigma^{-1}J''_{1d}+f_1J'_{1d} = f'_1g_2+(f_1+\eta f'_1-6f_2)\,g'_1, \qquad J_{1d}(0) = J_{1d}(\infty) = 0; \quad (3.27)$$

$$\sigma^{-1}K''_{1v}+f_1K'_{1v} = k_2F'_{0v}, \qquad K_{1v}(0) = K_{1v}(\infty) = 0; \quad (3.28)$$

$$\sigma^{-1}K''_{1d}+f_1K'_{1d} = f'_1k_2, \qquad K_{1d}(0) = K_{1d}(\infty) = 0. \quad (3.29)$$

Only (3.24), (3.25), and (3.28) need be integrated numerically, the displacement-speed functions being given by

$$G_{1d}(\eta) = \tfrac{1}{2}\eta g_1', \tag{3.30}$$

$$J_{1d}(\eta) = g_2 - \eta g_1', \tag{3.31}$$

$$K_{1d}(\eta) = k_2 - g_1. \tag{3.32}$$

For $\sigma = 0{\cdot}7$, numerical values are

$$\left.\begin{aligned} G_{1c}'(0) &= 0{\cdot}12811, & G_{1v}'(0) &= 0{\cdot}48449,\\ G_{1d}'(0) &= -0{\cdot}2479336, & J_{1d}'(0) &= 0{\cdot}133614,\\ K_{1v}'(0) &= -0{\cdot}31083, & K_{1d}'(0) &= -0{\cdot}213114.\end{aligned}\right\} \tag{3.33}$$

3.4. *Skin friction and heat transfer*

Substituting into (I, 4.13) shows that in unsymmetric flow the point of zero skin friction is shifted from the stagnation point for the basic inviscid flow to

$$s = s_0 = -R^{-\frac{1}{2}}U_{11}^{-1}[U_{20} + 1{\cdot}14111B_1'(0)/U_{11}^{\frac{1}{2}}] + O(R^{-1}). \tag{3.34}$$

The first term is simply the shift of the stagnation point in the outer flow, due to the displacement speed of the boundary layer. The second term is an additional shift within the boundary layer resulting from external vorticity. The shear producing this latter shift is

$$\tau_v = -1{\cdot}40652\mu\omega_0, \tag{3.35}$$

where ω_0 is the actual (dimensional) external vorticity on the stagnation streamline. The numerical factor means that through interaction with the boundary layer the shear associated with external vorticity is increased at the surface 41 % above its value in the free stream, compared with 77 % in axisymmetric flow. Neglecting the induced pressure gradient gives instead the decrease of 39 % predicted by Stuart (1959). The difference is of course that between the effects of displacement speed and displacement pressure.

It is convenient to shift the origin to the point of zero skin friction by setting $s' = s - s_0$. Then in terms of actual dimensional quantities the skin friction is given by

$$\begin{aligned}\tau = \rho\nu^{\frac{1}{2}}U_{11}^{\frac{3}{2}}s'[1{\cdot}232588 - 1{\cdot}913255(\nu/U_{11}L^2)^{\frac{1}{2}} + 5{\cdot}8466\nu^{\frac{1}{2}}\omega_0 U_{12}/U_{11}^{\frac{5}{2}}\\ + 1{\cdot}848882\nu^{\frac{1}{2}}(U_{21}/U_{11} - 2U_{12}U_{20}/U_{11}^2) + O(s'^2).\end{aligned} \tag{3.36}$$

Here L is the nose radius, s' is the true distance from the point of zero skin friction, and the actual surface speed in the outer flow is

$$(U_{11}s + U_{12}s^2 + \ldots) + \nu^{\frac{1}{2}}(U_{20} + U_{21}s + \ldots),$$

s being measured from the stagnation point of the basic inviscid flow. The first term is the classical result of Hiemenz. As in axisymmetric flow, convex curvature reduces the skin friction.

Substituting into (I, 4.14) gives the heat transfer at the point of zero skin friction for Prandtl number 0·7, in terms of dimensional quantities, as

$$\begin{aligned} q/k = (U_{11}/\nu)^{\frac{1}{2}}(T_{w0} - T_0)\,[0{\cdot}495867 - 0{\cdot}12811(\nu/U_{11}L^2)^{\frac{1}{2}} + 0{\cdot}89786\nu^{\frac{1}{2}}\omega_0 U_{12}/U_{11}^{\frac{5}{2}}\\ + 0{\cdot}247934\nu^{\frac{1}{2}}U_{21}/U_{11} - 0{\cdot}495867\nu^{\frac{1}{2}}U_{12}U_{20}/U_{11}^2]\\ + T_{w1}[0{\cdot}49820\omega_0/U_{11} - 0{\cdot}495867U_{20}/U_{11}^{\frac{1}{2}}] + O(\nu, s').\end{aligned} \tag{3.37}$$

Here the actual surface temperature is $(T_{w0}+T_{w1}s+\ldots)$. The corresponding expression at the inviscid stagnation point is obtained by replacing the third, fifth, sixth, and seventh numerical coefficients by 0·48449, $-$0·133614, $-$0·31083, and $+$0·213114 respectively (and s' by s).

Comparing with (3.36) shows that each of the second-order corrections proportional to $(T_{w0}-T_0)$ in (3.37) has the same sign as its counterpart for the skin friction, the relative effect being only from one-sixth to one-third as great.

4. Cusped leading edge

Consider a plane semi-infinite body having a cusped leading edge but otherwise analytic; this includes the standard problem of the semi-infinite flat plate. Because the leading edge is not analytic, the formal analysis of Part 1 is not strictly applicable. In fact, the inviscid velocity is generally infinite at the leading edge, and the actual flow presumably separates there.

If we restrict attention to the ideal angle of attack, for which the inviscid velocity is finite, the flow may remain attached. Classical boundary-layer theory has been applied to such cases with the understanding that local violation of the basic assumptions causes only local failure of the solution. It appears that this is true also of the second approximation, whereas Goldstein's (1960) and Imai's (1957) study of the flat plate shows that it is not true of the third approximation.

Consider either surface. The curvature, referred to some characteristic length L, has the form

$$\kappa(s) = \kappa_0 + O(s), \tag{4.1}$$

where κ_0 vanishes for the flat plate. The basic inviscid surface speed has the form

$$U_1(s,0) = U_{10} + O(s\log s), \tag{4.2}$$

where $U_{10} = 1$ for the flat plate or whenever the value at the leading edge is chosen as the reference speed U_r.

4.1. *First-order velocity and temperature*

Expand the first-order stream function in the Blasius series

$$\psi_1(s,N) = (2U_{10}s)^{\frac{1}{2}}\,[f_1(\eta)+O(s\log s)], \tag{4.3a}$$

where

$$\eta = (U_{10}/2s)^{\frac{1}{2}}\,N. \tag{4.3b}$$

Substituting into (I, 4.9) gives

$$f_1''' + f_1 f_1'' = 0, \quad f_1(0) = f_1'(0) = 0, \quad f_1'(\infty) = 1. \tag{4.4}$$

This is the Prandtl–Blasius problem, free of factors of 2 that appear in most references, thanks to the Falkner–Skan normalization. Numerical integration (Howarth 1938; Shanks 1953) yields

$$f_1''(0) = 0{\cdot}469600; \quad f_1(\eta) \sim \eta - \beta_1, \quad \beta_1 = 1{\cdot}21677. \tag{4.5}$$

The linearity of the energy equation is exploited by dividing its solution into the particular integral for an insulated body plus the complementary function for prescribed temperature. Let the dimensionless surface temperature be

$$T_w(s) = T_{w0} + O(s). \tag{4.6}$$

Then the Blasius series for temperature is*

$$t_1(s,N) = [H_1(0) - \tfrac{1}{2}m^2U_{10}^2] + m^2U_{10}^2 i_1(\eta) \\ + [T_{w0} - H_1(0) + \tfrac{1}{2}m^2U_{10}^2 - \tfrac{1}{2}m^2U_{10}^2 i_1(0)]\, g_1(\eta) + O(s), \quad (4.7)$$

where the first two terms alone given the solution for the insulated body. Substituting into (I, 4.10) gives

$$\sigma^{-1}g_1'' + f_1 g_1' = 0, \qquad g_1(0) = 1, \quad g_1(\infty) = 0; \qquad (4.8a)$$

$$\sigma^{-1}i_1'' + f_1 i_1' = -f_1''^2, \quad i_1'(0) = i_1(\infty) = 0. \qquad (4.8b)$$

The solutions can be found in terms of f_1, explicitly for $\sigma = 1{\cdot}0$ as

$$g_1(\eta) = 1 - f_1'(\eta), \quad i_1(\eta) = \tfrac{1}{2}(1 - f_1'^2) \qquad (4.9)$$

and otherwise by quadratures. Numerical values are

$$\left.\begin{aligned} -g_1'(0) &= 0{\cdot}469600, \quad \sigma = 1{\cdot}0, \\ &= 0{\cdot}41391, \quad \sigma = 0{\cdot}7; \\ i_1(0) &= 0{\cdot}50000, \quad \sigma = 1{\cdot}0, \\ &= 0{\cdot}41786, \quad \sigma = 0{\cdot}7. \end{aligned}\right\} \qquad (4.10)$$

4.2. *Second-order velocity*

Solving the problem for the flow due to displacement thickness will give

$$U_2(s,0) = U_{20} + O(s \log s). \qquad (4.11)$$

Then the second-order stream function is reduced to universal functions by setting

$$\psi_2(s,N) = (2s/U_{10})^{\frac{1}{2}}\, U_{20} F_{1d}(\eta) + 2s[\kappa_0 F_{2c}(\eta) + U_{10}^{-1} B_1'(0) F_{2v}(\eta)] + O(s^{\frac{3}{2}}). \quad (4.12)$$

Displacement speed is seen to be the dominant effect unless U_{20} vanishes, as it does for the semi-infinite plate. (For the finite plate, however, Kuo 1953 finds that $U_{20} = \beta_1/\pi$ approximately.) Substituting into (I, 4.11) gives for displacement speed the problem

$$F_{1d}''' + f_1 F_{1d}'' + f_1'' F_{1d} = 0, \quad F_{1d}(0) = F_{1d}'(0) = 0, \quad F_{1d}'(\infty) = 1, \qquad (4.13)$$

and for curvature and external vorticity

$$F_{2c}''' + f_1 F_{2c}'' - f_1' F_{2c}' + 2f_1'' F_{2c} = \eta f_1 f_1'' + \beta_1, \\ F_{2c}(0) = F_{2c}'(0) = 0, \quad F_{2c}''(\infty) = -1; \quad (4.14)$$

$$F_{2v}''' + f_1 F_{2v}'' - f_1' F_{2v}' + 2f_1'' F_{2v} = -\beta_1, \quad F_{2v}(0) = F_{2v}'(0) = 0, \quad F_{2v}''(\infty) = 1. \quad (4.15)$$

Kuo (1953) points out that the solution of (4.13) is

$$F_{1d}(\eta) = \tfrac{1}{2}(f_1 + \eta f_1'). \qquad (4.16)$$

The other problems have been integrated numerically. Thus

$$F_{1d}''(0) = 0{\cdot}70440, \quad F_{2c}''(0) = -3{\cdot}3910, \quad F_{2v}''(0) = 3{\cdot}1260. \qquad (4.17)$$

* Equation (14.54) of Schlichting (1960) is incorrect; it fails to reproduce the prescribed surface temperature.

The problem (4.15) for external vorticity has been the subject of prolonged controversy in the case of the semi-infinite flat plate, where $U_2(s,0)$ vanishes so that there is no effect of displacement speed. Li (1955) neglected the pressure gradient induced by interaction of displacement and external vorticity when he introduced the problem, but later (1956) corrected himself. Glauert (1957) and Ovchinnikov (1960) deny the existence of the induced pressure, and solve (4.15) with the term $-\beta_1$ omitted; according to Glauert's calculations this gives 0·795 for $F''_{2v}(0)$. A careful study by Murray (1961) confirmed the existence of the induced pressure, yielding a value of $F''_{2v}(0)$ that agrees with the present one to within one unit in the last place. The efficiency of the technique of inner and outer expansions is illustrated by the fact that Murray has to calculate the third term of the outer expansion, which is unnecessary here.

4.3. *Second-order temperature*

For simplicity, consider only strictly incompressible flow (dropping terms of order m^2). The second-order Blasius series for temperature is

$$\begin{aligned} t_2(s,N) &= (2U_{10}s)^{\frac{1}{2}} H'_1(0)\,[I_{2H}(\eta)-I_{2H}(0)\,G_{2H}(\eta)]+[T_{w0}-H_1(0)] \\ &\quad\times [(U_{20}/U_{10})\,G_{1d}(\eta)+(2s/U_{10})^{\frac{1}{2}}\{\kappa_0 G_{2c}(\eta)+U_{10}^{-1}B'_1(0)\,G_{2v}(\eta)\}]+O(s), \end{aligned} \tag{4.18}$$

where the G's are to be omitted for the insulated body and retained for the body with surface temperature prescribed by (4.6). As for the velocity field, the effect of displacement speed has the same dependence on s as the first approximation (but vanishes for the semi-infinite plate), whereas the other second-order effects are smaller by $s^{\frac{1}{2}}$.

Substituting into (I, 4.12) gives for displacement speed

$$\sigma^{-1}G''_{1d}+f_1G'_{1d} = -\tfrac{1}{2}(f_1+\eta f'_1)\,g'_1, \quad G_{1d}(0) = G_{1d}(\infty) = 0, \tag{4.19}$$

whose solution is

$$G_{1d}(\eta) = \tfrac{1}{2}\eta g'_1 \tag{4.20}$$

and for curvature, external vorticity, and stagnation enthalpy gradient,

$$\sigma^{-1}G''_{2c}+f_1G'_{2c}-j'_1G_{2c} = (\eta f_1-\sigma^{-1}-2F_{2c})\,g'_1, \quad G_{2c}(0) = G_{2c}(\infty) = 0; \tag{4.21}$$

$$\sigma^{-1}G''_{2v}+f_1G'_{2v}-f'_1G_{2v} = -2F_{2v}g'_1, \quad G_{2v}(0) = G_{2v}(\infty) = 0; \tag{4.22}$$

$$\sigma^{-1}G''_{2H}+f_1G'_{2H}-f'_1G_{2H} = 0, \quad G_{2H}(0) = 1, \quad G_{2H}(\infty) = 0; \tag{4.23}$$

$$\sigma^{-1}I''_{2H}+f_1I'_{2H}-f'_1I_{2H} = 0, \quad I'_{2H}(0) = 0, \quad I'_{2H}(\infty) = 1. \tag{4.24}$$

Numerical integration gives for $\sigma = 0{\cdot}7$

$$\left.\begin{aligned} G'_{1d}(0) = -0{\cdot}20696, \quad G'_{2c}(0) = 0{\cdot}70784, \quad G'_{2v}(0) = -0{\cdot}91117, \\ G'_{2H}(0) = -0{\cdot}57402, \quad I_{2H}(0) = 0{\cdot}86291. \end{aligned}\right\} \tag{4.25}$$

Aside from the different normalization, (4.22) agrees with equation (2.10) of Ovchinnikov (1960). His solution is therefore incorrect only because he uses an F_{2v} calculated without including the induced pressure.

4.4. *Skin friction, surface temperature, and heat transfer*

In physical variables the skin friction is

$$\tau = \rho U_{10}(\nu U_{10}/s)^{\frac{1}{2}}[0\cdot33206+0\cdot49809\nu^{\frac{1}{2}}U_{20}/U_{10} - 3\cdot3910(\nu s/U_{10})^{\frac{1}{2}}\kappa_0 - 3\cdot1260(\nu s/U_{10})^{\frac{1}{2}}\omega_0/U_{10}] + O(s^{\frac{1}{2}}, \nu^{\frac{3}{2}}), \quad (4.26)$$

where the actual surface speed in the outer flow is $(U_{10}+\nu^{\frac{1}{2}}U_{20}+\ldots)$. As in the previous examples, convex curvature decreases the skin friction. Murphy (1953) has found the same trend when the curvature is singular like $s^{-\frac{1}{2}}$, although this is disputed by Yen & Toba (1961).

External vorticity contributes

$$\tau_v = -3\cdot1260\mu\omega_0 \quad (4.27)$$

so that again the shear is increased by interaction with the boundary layer, to a value at the surface of more than three times that just outside the boundary layer. Neglecting the induced pressure gradient gives instead a 20% reduction.

For an insulated body with $\sigma = 0\cdot7$ the surface temperature is, in physical variables,

$$T_w(s) = T_0 + 1\cdot2203(\nu U_{10}s)^{\frac{1}{2}}(dT/dUn)_0 + O(s,\nu), \quad (4.28)$$

where T_0 is the stagnation temperature and $(dT/dUn)_0$ the external temperature gradient across the dividing streamline. For a positive gradient the surface temperature increases to prevent heat transfer to the body.

The heat transfer from a body at temperature $T_{w0}+O(s)$ with $\sigma = 0\cdot7$ is given, in physical variables, by

$$q/k = (U_{10}/\nu s)^{\frac{1}{2}}(T_{w0}-T_0)[0\cdot29268+0\cdot14634\nu^{\frac{1}{2}}U_{20}/U_{10} - 0\cdot70784(\nu s/U_{10})^{\frac{1}{2}}\kappa_0 - 0\cdot91117(\nu s/U_{10})^{\frac{1}{2}}\omega_0/U_{10}] - 0\cdot49533U_{10}(dT/dUn)_0 + O(s,\nu^{\frac{1}{2}}). \quad (4.29)$$

Each second-order effect has the same sign for the heat transfer as for the skin friction. The relative effect is always very nearly one-third as great, which may be taken as the second-order counterpart of Reynolds's analogy. The reduction in heat transfer due to external vorticity is several times greater than that predicted by Ovchinnikov (1960), who neglects the induced pressure gradient and finds the coefficient $-0\cdot340$ for $\sigma = 1\cdot0$ in place of the present $-0\cdot911$ for $\sigma = 0\cdot7$.

The coefficient multiplying the last term in (4.29) indicates that the heat transfer associated with an external temperature gradient is reduced at the surface to about half its value outside the boundary layer; the boundary layer tends to insulate the surface. Ovchinnikov calculates this coefficient as 0·454 for $\sigma = 1\cdot0$, and from the solution for $\sigma = 2\cdot0$ deduces that it varies approximately as $\sigma^{-0\cdot232}$; this gives 0·493 for $\sigma = 0\cdot7$, in close agreement with the present value.

4.5. *Leading-edge drag*

The skin friction (4.26) is integrable. Integrating formally would give for the drag of one surface, in physical variables,

$$D(s) = \rho U_{10}^2 s[0\cdot66411(\nu/U_{10}s)^{\frac{1}{2}} + 0\cdot99617\nu U_{20}U_{10}^{-\frac{3}{2}}s^{-\frac{1}{2}} - 3\cdot3910\nu\kappa_0/U_{10} - 3\cdot1260\nu\omega_0/U_{10}^2 + O(s^{\frac{1}{2}},\nu^{\frac{3}{2}})]. \quad (4.30a)$$

This is incorrect, because the solution is not valid near the leading edge. Imai (1957) has found the correct result for the semi-infinite flat plate, using a momentum control surface to avoid the region of invalidity. He shows that the above expression must be augmented by the constant

$$D_0 = \tfrac{1}{4}\pi\beta_1^2\mu U_{10} = 1{\cdot}163\mu U_{10}. \tag{4.30b}$$

From the gross point of view of boundary-layer theory this appears to be a force concentrated at the leading edge; it must actually be the result of increased skin friction in the vicinity of the leading edge, where $U_{10}s/\nu = O(1)$ and the boundary-layer approximation is invalid. It is analogous to the leading-edge drag of incompressible thin-airfoil theory (Jones & Cohen 1960). It is equal to the drag, in inviscid flow, of the solid parabola corresponding to the displacement thickness of the Blasius boundary layer.

Kuo (1953) has calculated the second approximation for a finite flat plate, and obtains for the coefficient of friction drag (of both surfaces)

$$C_D = 1{\cdot}328R^{-\frac{1}{2}} + 4{\cdot}12R^{-1}, \tag{4.31}$$

where R is the Reynolds number based on length. This agrees remarkably well with experiments of Janour (1951) down to $R = 10$. However, the leading-edge drag was not considered; adding it increases the second coefficient to 6·45. On the other hand, the constant 4·12 was estimated as the sum of a slowly convergent infinite series, and a re-examination shows that it can scarcely exceed 3. These two corrections tend to compensate each other, so that the second coefficient is approximately 5·3. This result is still in reasonable agreement with the experiments.

One might ask whether a concentrated force does not arise also from the trailing edge in this case. Reflexion suggests that it does, but that whereas the leading edge is exposed to the free-stream speed U, the velocities in a neighbourhood of order ν/U of the trailing edge (where the boundary-layer approximation is invalid) are reduced by the relatively thick boundary layer to order $R^{-\frac{1}{2}}U$. Hence the trailing-edge force is only a third-order effect.

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